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TikhonovFenichelReductions.jl: A systematic approach to geometric singular perturbation theory

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Summary

Natural systems often evolve on different characteristic time scales, with process rates differing substantially by orders of magnitude (Hek, 2010). To study such systems, singular perturbation theory provides a mathematical toolbox that translates this feature to ODE models and enables dimensionality reduction by focusing on a single characteristic time scale—typically the slow one governing the long-term behaviour (Wechselberger, 2020).

This separation of time scales is reflected by the presence of a small parameter $\varepsilon > 0$, which can be interpreted as the ratio between the two time scales. Roughly speaking, the dynamics in slow time $\tau = \varepsilon t$ can be approximated by replacing the fast part of the system with its steady state, so that the fast dynamics effectively appear instantaneous on the slow time scale.

More precisely, we obtain the *reduced system* (or *reduction* in short) as the limit

$$\begin{aligned} \dot{u} &= g(u,v) \\ \varepsilon \dot{v} &= h(u,v) \end{aligned} \quad \xrightarrow[\varepsilon \to 0]{} \quad \begin{aligned} \dot{u} &= g(u,v), \quad u \in \mathbb{R}^s \\ 0 &= h(u,v), \quad v \in \mathbb{R}^r \end{aligned} \tag{1}$$

with $\dot{} = \mathrm{d}/\mathrm{d}\tau$ as stated in Tikhonov's theorem (1952; Theorem 1.1 in Verhulst, 2007). The reduced system is defined on the so-called slow manifold $M_0 = \{(u,v) \,|\, h(u,v) = 0\}$.

For autonomous systems, the results of Fenichel (1979) allow a clear geometric interpretation—hence the name geometric singular perturbation theory (GSPT)—and guarantee that, for sufficiently small $\varepsilon > 0$, the reduction captures the behaviour of the full system.

In practice, GSPT can mitigate the realism-complexity trade-off, as modellers can work with the reduction instead of the full system, which is often more tractable. Because the reduction remains embedded in the original system through the time scale separation, the original interpretability is retained. This allows one to follow the modelling paradigm: start with a complex but realistic model, reduce later.

The algebraic approach recently developed by Goeke & Walcher (and colleagues) allows one to work with GSPT systematically and is coordinate-free, i.e., we do not rely on the standard form (1) with a time scale separation of components (Goeke, 2013; Goeke et al., 2015; Goeke & Walcher, 2013, 2014). Instead, we consider systems of the form

$$x' = f^{(0)}(x) + \varepsilon f^{(1)}(x) + \mathcal{O}(\varepsilon^2), \quad x \in \mathbb{R}^n, \tag{2}$$

where $f^{(0)}$ contains the fast processes and $f^{(1)}$ contains the slow ones. Note that system (1) can also be written in fast time t as

$$\begin{aligned} u' &= \varepsilon g(u,v) \\ v' &= h(u,v) \end{aligned} \tag{3}$$

where $' = \mathrm{d}/\mathrm{d}t$. Thus, we can translate (2) into (3) by setting $x = (u, v)^{\mathrm{T}}$, $f^{(0)} = (0, h)^{\mathrm{T}}$ and $f^{(1)} = (g, 0)^{\mathrm{T}}$. However, the reverse direction requires a suitable coordinate transformation, which is not always feasible. Fortunately, this transformation is not explicitly needed to obtain the reduced system, which can be computed using (4).

The main benefit of this framework is, however, that it allows one to find all critical parameters admitting a reduction for polynomial or rational ODE systems of the form (2). Thus, modellers do not have to rely on prior knowledge to find suitable time scale separations as they can systematically consider all of them. This can be achieved by evaluating necessary conditions for the existence of a reduction (Goeke, 2013; Goeke et al., 2015). `TikhonovFenichelReductions.jl` is a `Julia` (Bezanson et al., 2017) package implementing this approach for polynomial ODE systems. Apelt & Liebscher (2025) provide a showcasing example and more detailed explanations.

Statement of need

The ad-hoc approach to singular perturbation theory requires prior knowledge of a suitable time scale separation and substantial mathematical effort to compute the reduction. In contrast, the algebraic framework developed by Goeke & Walcher systematically yields *all* possible reductions for a given polynomial ODE system, enabling modellers to consider several scenarios defined by different time scale separations (Goeke et al., 2015). `TikhonovFenichelReductions.jl` makes the required computations easily accessible.

To the author's knowledge, no publicly available implementation of this theory currently exists. However, there is an implementation of a related approach for computing invariant manifolds by Roberts (n.d., 1997).

Software design

`TikhonovFenichelReductions.jl` is implemented in `Julia` due to its flexibility, the use of multiple dispatch and the availability of the feature-rich computer algebra system `Oscar.jl` (Decker et al., 2025; *OSCAR*, 2026).

Core features are the search for critical parameters admitting a reduction, so-called *Tikhonov-Fenichel Parameter Values (TFPVs)*, and the computation of the corresponding reduced systems. Crucially, this requires various computations with multiple symbolic representations (e.g., different polynomial rings, rational function fields and matrix spaces) and therefore parsing of data between the corresponding types in `Oscar.jl`, which `TikhonovFenichelReductions.jl` performs hidden away from the user. Thus, the user mostly works in an object-oriented manner with the types and methods provided.

The package is essentially designed around two types: `ReductionProblem`, which constructs all symbolic data types needed for the search of TFPVs, and `Reduction`, which holds all relevant information for the reduced system and the steps required to compute it. The latter also contains the reduced system and other information parsed to the appropriate types from `Oscar.jl`, that can be further used, e.g., for a symbolic analysis.

Detailed explanations and examples are provided by Apelt & Liebscher (2025) and in the documentation. The core features are summarized in the following.

Finding TFPVs

The package provides a method to obtain all possible TFPVs implicitly by computing a Gröbner basis and one to find *slow-fast separations of rates*, i.e., TFPVs with some parameters set to zero. Although the former method is an exhaustive search, the latter is usually better suited in practice as it yields the TFPVs one is typically interested in explicitly, is more efficient, and directly outputs the corresponding slow manifolds (implicitly as affine varieties in phase space). In both cases, one utilizes the correspondence between algebraic and geometric properties of the slow manifold, which renders this an algebraic approach to GSPT.

Computing reductions

Computing a reduction for a slow-fast separation of rates as in Theorem 1 in Goeke & Walcher (2014) requires essentially two steps. Let s be the dimension of the reduced system and $r = n - s$. First, we need to provide a parametric representation of the slow manifold, which is given as an irreducible component of the affine variety $\mathcal{V}(f^{(0)})$. Then, we need to find a product decomposition $f^{(0)} = P\psi$, where P is a $n \times r$ matrix of rational functions and ψ is a vector of polynomials locally satisfying $\mathcal{V}(\psi) = \mathcal{V}(f^{(0)})$ and rank $P =$ rank $D\psi = r$.

With this, the reduced system in the sense of Tikhonov is given by

$$x' = \left[1_n - P(x)(D\psi(x)P(x))^{-1}D\psi(x)\right] f^{(1)}(x). \tag{4}$$

For convenience, the packages provides multiple heuristics, which automate the steps required to compute a reduction and allows bulk computation of multiple reductions at once.

Integration with the `Julia` ecosystem

The input system can be given as a reaction network defined with `Catalyst.jl` (Loman et al., 2023). Because the reduced systems are represented using types from `Oscar.jl`, the latter's functions can be used to aid the symbolic analysis. `Julia`'s support for metaprogramming allows to perform further tasks such as a numerical analysis without having to copy or parse code (see, e.g., `TFRSimulations.jl`). For convenience, there are several methods for displaying the output, including printing as LaTeX source code via `Latexify.jl`.

Research impact statement

Time scale separations are widely used in various areas of mathematical modelling (Wechselberger, 2020). However, as far as the author is aware, the systematic approach due to Goeke & Walcher seems to be scarcely adopted, even though it comes with many advantages compared to the ad-hoc approach. This package aims to make the theory more accessible and convenient to use.

In the field of mathematical ecology (which the author is most familiar with), the theory was successfully used by Kruff et al. (2019) and more recently by Apelt & Liebscher (2025). For the model introduced in the former, the application was relatively straightforward, but more complex models as in the latter require some more work. In particular, the method for finding TFPVs that relies on the computation of a Gröbner basis may fail for complex systems. In this case, `TikhonovFenichelReductions.jl` may enable and simplify the search for the most common TFPVs. Given the ubiquity of time scale separation techniques in this field alone (see, e.g., Abbott et al., 2020; Poggiale & Auger, 2004; Revilla, 2015), the package can be a potentially useful tool for modellers.

AI usage disclosure

No generative AI tools were used in the development of this software, the writing of this manuscript, or the preparation of supporting materials.

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