








FEniCS-arclength: A numerical continuation package in FEniCS for nonlinear problems in solid mechanics

Peerasait Prachaseree ¹, Saeed Mohammadzadeh ², Berkin Dortdivanlioglu ^{3,4}, and Emma Lejeune ¹✉

1 Department of Mechanical Engineering, Boston University, Massachusetts, the United States of America **2** Department of Systems Engineering, Boston University, Massachusetts, the United States of America **3** Department of Civil, Architectural and Environmental Engineering, The University of Texas at Austin, Austin, the United States of America **4** Oden Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, the United States of America ✉ Corresponding author

DOI: [10.21105/joss.05727](https://doi.org/10.21105/joss.05727)

Software

- [Review](#) 
- [Repository](#) 
- [Archive](#) 

Editor: [Prashant K Jha](#) 

Reviewers:

- [@w-luo-academic](#)
- [@kingyin3613](#)

Submitted: 29 June 2023

Published: 02 February 2024

License

Authors of papers retain copyright and release the work under a Creative Commons Attribution 4.0 International License ([CC BY 4.0](https://creativecommons.org/licenses/by/4.0/)).

Summary

FEniCS-arclength is a package that implements the numerical arclength continuation method, following the pioneering work in developing the method ([Crisfield, 1981](#); [Riks, 1979](#)), as a nonlinear solver in the open source finite element software framework FEniCS ([Alnæs et al., 2015](#); [Logg et al., 2012](#)). The arclength method is a robust nonlinear solver that is capable of capturing complex equilibrium paths including unstable states in solid mechanics problems with structural instabilities. Such structural instabilities are induced by geometric nonlinearities, whereas material and contact nonlinearities can also be accommodated in the framework.

FEniCS ([Alnæs et al., 2015](#); [Logg et al., 2012](#)) is an open-source software that solves partial differential equations (PDEs) through finite element analysis (FEA). Building on the versatile FEniCS platform, researchers have created both detailed tutorials ([Bleyer, 2018](#)) and software packages ([Finsberg, 2019](#); [Rodriguez et al., 2019](#)) to facilitate research in specific application domains. For example, there are multiple open-source software libraries designed to increase the existing capabilities of FEniCS ([Farrell et al., 2016](#); [Kamensky & Bazilevs, 2019](#); [Mitusch et al., 2019](#)). Building on this established precedent, we have implemented our arclength solver as an add-on library on top of FEniCS, and also demonstrated usage of our solver via multiple documented examples involving nonlinear continuum and beam finite elements at large deformations. Notably, our intent is to keep the solver usage similar to already established FEniCS solvers to enable off-the-shelf implementation within FEniCS workflows. To this end, our arclength solver can be integrated into existing FEniCS code that is based on a Newton solver just by changing the choice of solver. Furthermore, our code and documentation are organized such that more advanced users can make custom modifications as needed, for example pinpointing and branch switching algorithms ([Wriggers et al., 1988](#)).

In brief, our implementation follows a recently published method ([Kadapa, 2021](#)) adopting an extrapolated predictor scheme using the previously converged two solution steps, and solves the augmented arclength matrix equation as the corrector scheme. The main advantage of the extrapolation prediction scheme is in its ability to determine the forward direction of the equilibrium path without doing extra computations such as evaluating the determinant of the stiffness matrix ([Crisfield, 1981](#)), although calculating the sign of the determinant can be simple depending on the type of linear solver (i.e., the sign of the determinant can be extracted during numerical factorization). For displacement-controlled arclength problems, similar to a multifreedom constraint problem, we introduced a constraint matrix to augment the prescribed non-zero Dirichlet boundary conditions and the free degrees of freedoms to solve the arclength equations ([Batoz & Dhatt, 1979](#); [Verhoosel et al., 2009](#)). More details

about the theory, implementation, usage of our arclength solver, and general mathematical preliminaries can be found in our extensive [ReadTheDocs documentation](#).

Statement of Need

Materials and structures that undergo large deformation often exhibit fascinating mechanical instabilities due to their geometrically nonlinear behavior. Recently, there has been a renewed interest in studying these mechanical instabilities in engineered architected materials ([Medina et al., 2020](#)), and in leveraging our understanding of mechanics to interpret the non-linear behavior of natural materials ([J.-H. Lee et al., 2021](#)) and the elastocapillary behavior of soft solids ([Dortdivanlioglu & Javili, 2022](#)). In these applied mechanics problems, the finite element method is often used to computationally simulate these complex structures, particularly when geometry and/or material behavior are complex and consequently previously derived and validated analytical solutions are not applicable. In the solutions of geometrically nonlinear problems with finite element analysis, the equilibrium path of the structure is traced either in a force-controlled scheme (i.e., prescribed incremental Neumann boundary conditions) or displacement-controlled scheme (i.e., prescribed incremental non-zero Dirichlet boundary conditions). However, in extreme cases, such as in the case of snap-back or snap-through instabilities, standard Newton solver based force/displacement-controlled schemes fail. To address this shortcoming of standard Newton solvers, various nonlinear solvers have been proposed based on the common approach of adding an extra constraint equation on top of the Newton solver ([Leon et al., 2012](#)). In particular, the arclength method addresses this challenge through an arclength constraint equation where an incremental load or displacement factor at each load iteration, an additional unknown, is modified. Various arclength-type solvers are available in commercial FEA software (e.g., see Riks method in ABAQUS ([Smith, 2009](#))) to capture the full equilibrium path. The goal of this work is to provide the research community with an open-source and robust arclength solver that can readily be applied to a myriad of nonlinear solid mechanics problems.

Examples of FEniCS Arclength

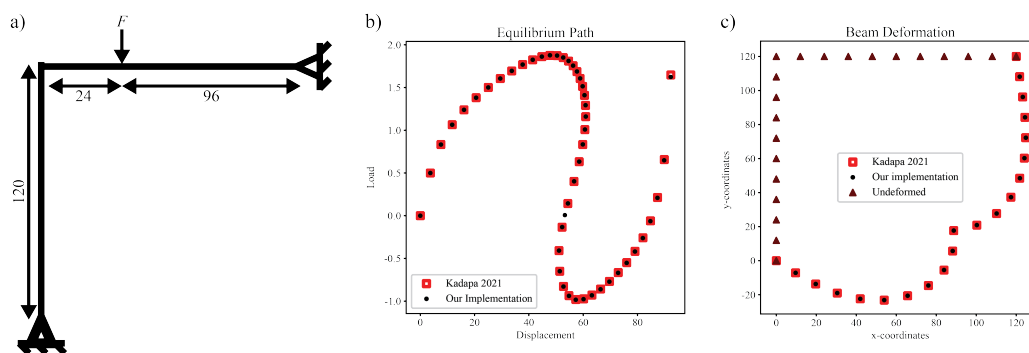


Figure 1: A demonstration of our arclength solver via the snap-back instability behavior of Lee's frame, a popular benchmark problem in nonlinear structural mechanics: a) The problem definition for Lee's frame; b) The equilibrium path from our solver compared to the literature ([Kadapa, 2021](#)); c) The final deformed configuration from our solver compared to the literature ([Kadapa, 2021](#)).

In [Figure 1](#), we show an example of our arclength implementation for solving the Lee's frame problem ([S.-L. Lee et al., 1968](#)), a popular benchmark problem in nonlinear structural mechanics. In this problem, the frame is discretized using 2D Simo-Reissner beams ([Simo & Vu-Quoc, 1986](#)) with mixed elements. The elements have linear Lagrange interpolation functions for both displacement and rotation, and use reduced integration to avoid locking. [Figure 1a](#) gives

an overview of the Lee's frame problem and illustrates the equilibrium path (Figure 1b) and the deformed configuration (Figure 1c) in comparison with the literature (Kadapa, 2021). The Lee's frame example demonstrates that our solver is able to trace unstable states, where traditional Newton solvers would fail. Additional validation examples on problems in nonlinear structural mechanics with beam elements (compared to solutions from Kadapa (2021)), and bilayer wrinkling with continuum elements (compared to analytical solutions (Allen, 1969; Budday et al., 2017; Cao & Hutchinson, 2012)) can be found in the examples folder of the GitHub repository, and in the ReadTheDocs documentation. These examples can be easily run through Google Colab (Google Colaboratory, 2017) with FEM on Colab (FEM on Colab, 2021). Critically, we demonstrated that our solver is able to trace the complex equilibrium paths of popular benchmark problems such as 215° and 180° arches (see Table 1 for all the provided examples in our documentation). In addition to validation problems, our documentation also contains the solver's API, example usage of our solver for continuum problems, and example usage of our displacement controlled arclength scheme.

Table 1: A list of all available examples in our GitHub Repository and ReadTheDocs documentation. Examples marked with (×) are compared with solutions from the literature obtained from Kadapa (2021). Examples with (★) are compared with solutions from Allen (1969); Budday et al. (2017); and Cao & Hutchinson (2012).

Force-controlled Scheme
<i>Continuum elements</i>
▪ Lee's Frame (Continuum Elements)
<i>Beam elements</i>
▪ Lee's Frame ×
▪ 215° arch ×
▪ 180° arch ×
▪ 3D Helical Beam
Displacement-controlled Scheme
<i>Continuum Elements</i>
▪ Bilayer Wrinkling ★
<i>Beam Elements</i>
▪ Random Fiber Network

Acknowledgements

This work was made possible with funding through the Boston University David R. Dalton Career Development Professorship, the Hariri Institute Junior Faculty Fellowship, the Haythornthwaite Foundation Research Initiation Grant, and the National Science Foundation Grant CMMI-2127864. This support is gratefully acknowledged.

References

- Allen, H. G. (1969). *Analysis and design of structural sandwich panels*. Pergamon.
- Alnæs, M., Blechta, J., Hake, J., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M. E., & Wells, G. N. (2015). The FEniCS project version 1.5. *Archive of Numerical Software*, 3(100).
- Batoz, J.-L., & Dhatt, G. (1979). Incremental displacement algorithms for nonlinear problems. *International Journal for Numerical Methods in Engineering*, 14(8), 1262–1267. <https://doi.org/10.1002/nme.1620140811>
- Bleyer, J. (2018). *Numerical tours of computational mechanics with FEniCS*. <https://comet-fenics.readthedocs.io>; Zenodo. <https://doi.org/10.5281/zenodo.1287832>

- Budday, S., Andres, S., Walter, B., Steinmann, P., & Kuhl, E. (2017). Wrinkling instabilities in soft bilayered systems. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 375(2093), 20160163. <https://doi.org/10.1098/rsta.2016.0163>
- Cao, Y., & Hutchinson, J. W. (2012). Wrinkling Phenomena in Neo-Hookean Film/Substrate Bilayers. *Journal of Applied Mechanics*, 79(3), 031019. <https://doi.org/10.1115/1.4005960>
- Crisfield, M. A. (1981). A fast incremental/iterative solution procedure that handles “snap-through.” *Computers & Structures*, 13(1), 55–62. [https://doi.org/10.1016/0045-7949\(81\)90108-5](https://doi.org/10.1016/0045-7949(81)90108-5)
- Dortdivanlioglu, B., & Javili, A. (2022). Plateau Rayleigh instability of soft elastic solids. Effect of compressibility on pre and post bifurcation behavior. *Extreme Mechanics Letters*, 55, 101797. <https://doi.org/10.1016/j.eml.2022.101797>
- Farrell, P. E., Beentjes, C. H., & Birkisson, Á. (2016). The computation of disconnected bifurcation diagrams. *arXiv Preprint arXiv:1603.00809*.
- FEM on colab. (2021). <https://fem-on-colab.github.io/index.html>
- Finsberg, H. N. T. (2019). Pulse: A python package based on FEniCS for solving problems in cardiac mechanics. *Journal of Open Source Software*, 4(41), 1539. <https://doi.org/10.21105/joss.01539>
- Google colab. (2017). <https://colab.google/>
- Kadapa, C. (2021). A simple extrapolated predictor for overcoming the starting and tracking issues in the arc-length method for nonlinear structural mechanics. *Engineering Structures*, 234, 111755. <https://doi.org/10.1016/j.engstruct.2020.111755>
- Kamensky, D., & Bazilevs, Y. (2019). tIGAr: Automating isogeometric analysis with FEniCS. *Computer Methods in Applied Mechanics and Engineering*, 344, 477–498. <https://doi.org/10.1016/j.cma.2018.10.002>
- Lee, J.-H., Park, H. S., & Holmes, D. P. (2021). Elastic instabilities govern the morphogenesis of the optic cup. *Phys. Rev. Lett.*, 127, 138102. <https://doi.org/10.1103/PhysRevLett.127.138102>
- Lee, S.-L., Manuel, F. S., & Rossow, E. C. (1968). Large deflections and stability of elastic frame. *Journal of the Engineering Mechanics Division*, 94(2), 521–548. <https://doi.org/10.1061/JMCEA3.0000966>
- Leon, S. E., Paulino, G. H., Pereira, A., Menezes, I. F. M., & Lages, E. N. (2012). A Unified Library of Nonlinear Solution Schemes. *Applied Mechanics Reviews*, 64(4), 040803. <https://doi.org/10.1115/1.4006992>
- Logg, A., Mardal, K.-A., & Wells, G. (2012). *Automated solution of differential equations by the finite element method: The FEniCS book* (Vol. 84). Springer Science & Business Media.
- Medina, E., Farrell, P. E., Bertoldi, K., & Rycroft, C. H. (2020). Navigating the landscape of nonlinear mechanical metamaterials for advanced programmability. *Phys. Rev. B*, 101, 064101. <https://doi.org/10.1103/PhysRevB.101.064101>
- Mitsch, S. K., Funke, S. W., & Dokken, J. S. (2019). Dofin-adjoint 2018.1: Automated adjoints for FEniCS and firedrake. *Journal of Open Source Software*, 4(38), 1292. <https://doi.org/10.21105/joss.01292>
- Riks, E. (1979). An incremental approach to the solution of snapping and buckling problems. *International Journal of Solids and Structures*, 15(7), 529–551. [https://doi.org/10.1016/0020-7683\(79\)90081-7](https://doi.org/10.1016/0020-7683(79)90081-7)

- Rodriguez, M. A., Augustin, C. M., & Shadden, S. C. (2019). FEniCS mechanics: A package for continuum mechanics simulations. *SoftwareX*, 9, 107–111. <https://doi.org/10.1016/j.softx.2018.10.005>
- Simo, J. C., & Vu-Quoc, L. (1986). A three-dimensional finite-strain rod model. Part II: Computational aspects. *Computer Methods in Applied Mechanics and Engineering*, 58(1), 79–116. [https://doi.org/10.1016/0045-7825\(86\)90079-4](https://doi.org/10.1016/0045-7825(86)90079-4)
- Smith, M. (2009). *ABAQUS/standard user's manual, version 6.9*. Dassault Systèmes Simulia Corp.
- Verhoosel, C. V., Remmers, J. J. C., & Gutiérrez, M. A. (2009). A dissipation-based arc-length method for robust simulation of brittle and ductile failure. *International Journal for Numerical Methods in Engineering*, 77(9), 1290–1321. <https://doi.org/10.1002/nme.2447>
- Wriggers, P., Wagner, W., & Miehe, C. (1988). A quadratically convergent procedure for the calculation of stability points in finite element analysis. *Computer Methods in Applied Mechanics and Engineering*, 70, 329–347. [https://doi.org/10.1016/0045-7825\(88\)90024-2](https://doi.org/10.1016/0045-7825(88)90024-2)